Runge-Kutta Methods of order 2

(Midpoint method)

Recall:

Taylor methods of order 2:

$$\boxed{\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h \underbrace{T^{(2)}(t_i, w_i)}_{f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)} \end{cases}}$$

(error = $O(h^2)$)

Idea: Find a_1, α_1, β_1 s.t.

$$T^{(2)}(t_i, w_i) \approx a_1 f(t + \alpha_1, y + \beta_1)$$

and error stays $O(h^2)$

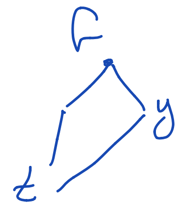
The derivation requires 2 Ingredients

- Ingredient I : Taylor polynomials for functions of 2 variables :

$$
\begin{aligned}
&f(t+\alpha_1, y+\beta_1) \\
&= f(t,y) + \begin{bmatrix} \alpha_1 & \beta_1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial t} \\[2mm] \dfrac{\partial f}{\partial y} \end{bmatrix}\Bigg|_{(t,y)} \\
&\quad \underbrace{+ \begin{bmatrix} \alpha_1 & \beta_1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 f}{\partial t^2} & \dfrac{\partial^2 f}{\partial t \partial y} \\[2mm] \dfrac{\partial^2 f}{\partial t \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}\Bigg|_{(\xi,\mu)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}}_{\text{error term "}R_1\text{" or "}E_1\text{"}}
\end{aligned}
\qquad (1)
$$

where (ξ,μ) between (t,y) & $(t+\alpha_1, y+\beta_1)$

- Ingredient II: Chain Rule



$$f'(t,y) = \frac{df}{dt}(t, y(t))$$

$$= \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\cdot \underbrace{\frac{dy}{dt}}_{y' = f}$$

but $T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y)$

so
$$T^{(2)}(t,y) = f + \frac{h}{2}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\cdot f\right] \tag{2}$$

Rewriting (1) & multiplying it by a_1:

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1\frac{\partial f}{\partial y}(t,y) + a_1 R_1 \tag{3}$$

Since we want $T^{(2)}(t,y) \approx a_1 f(t+\alpha_1, y+\beta_1)$

we just match coefficients of f, $\frac{\partial f}{\partial t}$ & $\frac{\partial f}{\partial y}$ in (2) & (3) And we solve for a_1, α_1, β_1 :

$$\Rightarrow \boxed{a_1 = 1}, \boxed{\frac{h}{2} = a_1 \alpha_1}, \boxed{\frac{h}{2} \cdot f(t,y) = a_1 \beta_1}$$

So now :

$$f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right) = T^{(2)}(t,y)$$
$$+ \underbrace{R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right)}_{\left[\frac{h}{2} \quad \frac{h}{2} f(t,y)\right] \begin{bmatrix} \frac{\partial^2 f}{\partial t^2} & \frac{\partial^2 f}{\partial t \partial y} \\ \frac{\partial^2 f}{\partial t \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} \frac{h}{2} \\ \frac{h}{2} f(t,y) \end{bmatrix}}$$

$\approx O(h^2)$ if all partials are bounded.

So we replace $T^{(2)}(t,y)$ by $f(t + h/2, y + h/2\, f(t,y))$

to get:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right) \end{cases}$$

for $i = 0, \dots, N-1$

Runge-Kutta method of order 2

also known as: Midpoint method

Another $O(h^2)$ method is given by

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}\left[\underbrace{f(t_i, w_i)}_{y' \text{ at } (t_i, w_i)} + \underbrace{f(t_{i+1}, w_i + hf(t_i, w_i))}_{\text{est. of } y' \text{ at } (t_{i+1},\, w_i + hf(t_i, w_i))}\right] \end{cases}$$

modified Euler Method

$w_i + hf(t_i, w_i)$: 1st order approx of w_{i+1}

$\approx$ Average of derivatives/slopes

Idea behind this modification:

$w_i + h f(t_i, w_i) =: \tilde{w}_{i+1}$ is the estimate from Euler's method of y_{i+1}.

Rather than use $\tilde{w}_{i+1}$ directly, we plug it back into

$f(t_{i+1}, \tilde{w}_{i+1}) \approx$ estimate of slope at t_{i+1}

and average it with $f(t_i, w_i)$, to get

$$\begin{cases} w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, \tilde{w}_{i+1})\right] \\ \text{where } \tilde{w}_{i+1} = w_i + h f(t_i, w_i) \end{cases}$$

Example of Higher order RK method:

RK order 4:

est of y' at $t_i + h/2$

$$
\begin{cases}
w_0 = \alpha \\
k_1 = hf(t_i, w_i) \\
k_2 = hf\left(t_i + h/2,\ \underbrace{w_i + \tfrac{1}{2}k_1}\right) \\
k_3 = hf\left(t_i + h/2,\ \underbrace{w_i + \tfrac{1}{2}k_2}\right) \\
k_4 = hf\left(\underline{t_{i+1}},\ w_i + k_3\right) \\
w_{i+1} = w_i + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{cases}
$$

$$
\boxed{\begin{array}{l} \text{Error} = O(h^4) \\ \quad (\text{provided } y \in C^{(5)}) \end{array}}
$$

Summary of our numerical methods for IVP's:

$$\begin{cases} y' = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$

(I) Euler's method $O(h)$

$$\boxed{\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h f(t_i, w_i) \end{aligned}}$$

(II) Higher order Taylor methods $O(h^n)$

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h\, T^{(n)}(t_i, w_i)$$

where $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$

(III) Runge-Kutta methods

(IIIa) Order 2 $(O(h^2))$

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right)$$

(III b) Order 4 $(O(h^4))$

(See before)

(IV) Other modifications

e.g. modified Euler

$$w_0 = \alpha$$
$$w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, \tilde{w}_{i+1})\right)$$

where

$$\tilde{w}_{i+1} = w_i + hf(t_i, w_i)$$